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A Theorem on the Generalized Stieltjes Transform and Its Applications

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This paper presents a proof of Parseval–Goldstein type theorem involving the generalized Stieltjes transform and the classical Laplace transform. The theorem is then shown to yield a number of results about the Widder potential transform and the Laguerre transform. Some illustrative examples are also given.

INTRODUCTION AND DEFINITIONS

Goldstein [2] introduced the following Parseval-type theorem

$$\int_0^\infty f(x)\,\mathscr{L}[g(y); x]\,dx = \int_0^\infty g(y)\,\mathscr{L}[f(x); y]\,dy \tag{1}$$

for the classical Laplace transform

$$\mathscr{L}[f(x); y] = \int_0^\infty e^{-xy} f(x)\,dx. \tag{2}$$

Subsequently, the author [8] gave the following Parseval–Goldstein type theorem

$$\int_0^\infty \mathscr{L}[f(u); x]\,\mathscr{L}[g(y); x]\,dx = \int_0^\infty g(y)\,\mathscr{S}[f(u); y]\,dy \tag{3}$$

involving the Laplace transform and the Stieltjes transform

$$\mathscr{S}[f(x); y] = \int_0^\infty \frac{f(x)}{x+y}\,dx. \tag{4}$$

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There are numerous analogous results in the literature on integral transforms (see, for instance, [3–5, 9]).

The objective of this paper is to establish a Parseval–Goldstein type theorem, which is a generalization of (3), involving the Laplace transform and the generalized Stieltjes transform

$$\mathscr{S}_\rho[f(x); y] = \int_0^\infty \frac{f(x)}{(x+y)^\rho}\,dx. \tag{5}$$

We present some results on the Fourier sine transform

$$\mathscr{F}_s[f(x); y] = \int_0^\infty \sin(xy)\,f(x)\,dx, \tag{6}$$

on the Fourier cosine transform

$$\mathscr{F}_c[f(x); y] = \int_0^\infty \cos(xy)\,f(x)\,dx, \tag{7}$$

on the Widder potential transform [6, 7]

$$\mathscr{P}[f(x); y] = \int_0^\infty \frac{xf(x)}{x^2+y^2}\,dx, \tag{8}$$

on the Laguerre transform

$$\mathscr{T}_\alpha[f(x); n] = \int_0^\infty e^{-x}x^\alpha \mathscr{L}_n^{(\alpha)}(x)\,f(x)\,dx, \tag{9}$$

where

$$\mathscr{L}_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) \qquad (n = 0, 1, 2, ..., \alpha > -1) \tag{10}$$

are the associated Laguerre polynomials, and on the Mellin transform

$$\mathscr{M}[f(x); y] = \int_0^\infty x^{y-1}f(x)\,dx. \tag{11}$$

For definitions of special functions that are used in the paper, the reader is referred to [1].

If we write $\mathscr{X}[f(x); y] = F(y)$, where $\mathscr{X}$ is any one of the integral transforms listed above, we mean the $\mathscr{X}$-transform of $f(x)$ exists and is equal to $F(y)$.

THE MAIN THEOREM AND ITS COROLLARIES

Throughout the remainder of this paper it is assumed that all integrals involved converge absolutely. The following result will be required in our investigation:

LEMMA. *We have*

$$\mathscr{L}[u^{\rho-1}\mathscr{L}[f(x); u]; y] = \Gamma(\rho)\,\mathscr{S}_\rho[f(x); y] \qquad (\operatorname{Re}\rho > 0). \tag{12}$$

Proof. From the definition (2) of the Laplace transform we obtain

$$\mathscr{L}[u^{\rho-1}\mathscr{L}[f(x); u]; y] = \int_0^\infty u^{\rho-1}e^{-uy}\left\{\int_0^\infty e^{-ux}f(x)\,dx\right\}du. \tag{13}$$

Changing the order of integration, which is permissible by the absolute convergence of the integrals involved, we find from (13) that

$$\begin{aligned}\mathscr{L}[u^{\rho-1}\mathscr{L}[f(x); u]; y] &= \int_0^\infty f(x)\left\{\int_0^\infty u^{\rho-1}e^{-(x+y)u}\,du\right\}dx\\ &= \int_0^\infty f(x)\,\mathscr{L}[u^{\rho-1}; x+y]\,dx\\ &= \Gamma(\rho)\int_0^\infty \frac{f(x)}{(x+y)^\rho}\,dx. \end{aligned} \tag{14}$$

Now the assertion (12) follows from (5) and (14). This completes the proof of the lemma.

We now state the main theorem:

MAIN THEOREM. *We have*

$$\begin{aligned}&\int_0^\infty x^{\rho-1}\mathscr{L}[f(u); x]\,\mathscr{L}[g(y); x]\,dx\\ &\qquad = \Gamma(\rho)\int_0^\infty g(y)\,\mathscr{S}_\rho[f(u); y]\,dy \qquad (\operatorname{Re}\rho > 0).\end{aligned} \tag{15}$$

Proof. From the definition (2) of the Laplace transform we obtain

$$\begin{aligned}&\int_0^\infty x^{\rho-1}\mathscr{L}[f(u); x]\,\mathscr{L}[g(y); x]\,dx\\ &\qquad = \int_0^\infty x^{\rho-1}\mathscr{L}[f(u); x]\left\{\int_0^\infty e^{-xy}g(y)\,dy\right\}dx.\end{aligned} \tag{16}$$

Changing the order of integration, which is permissible by the absolute convergence of the integrals involved, we find from (16) that

$$\begin{aligned}\int_0^\infty & x^{\rho-1}\mathscr{L}[f(u);x]\,\mathscr{L}[g(y);x]\,dx\\ &=\int_0^\infty g(y)\left\{\int_0^\infty x^{\rho-1}e^{-xy}\mathscr{L}[f(u);x]\,dx\right\}dy\\ &=\int_0^\infty g(y)\,\mathscr{L}[x^{\rho-1}\mathscr{L}[f(u);x];y]\,dy. \qquad (17)\end{aligned}$$

Now the assertion (15) follows from (12) and (17). This completes the proof of the theorem under the hypothesis stated.

Remark. If we use the convolution property of the Laplace transform in (15), we obtain

$$\int_0^\infty x^{\rho-1}\mathscr{L}[(f*g)(y);x]\,dx=\Gamma(\rho)\,\mathscr{S}_\rho[f(u);y]\,dy \qquad (\operatorname{Re}\rho>0). \qquad (18)$$

COROLLARY. *We have*

$$\int_0^\infty f(u)\,\mathscr{S}_\rho[g(y);u]\,du=\int_0^\infty g(y)\,\mathscr{S}_\rho[f(u);y]\,dy \qquad (\operatorname{Re}\rho>0). \qquad (19)$$

Proof. Note that the left-hand side of Eq. (15) is symmetrical with respect to the functions f and g. This fact implies the assertion (19).

COROLLARY. *We have*

$$\begin{aligned}&\mathscr{S}_\nu[x^{\rho-1}\mathscr{L}[f(u);x];z]\\ &\quad=\frac{\Gamma(\rho)}{\Gamma(\nu)}\mathscr{L}[y^{\nu-1}\mathscr{S}_\rho[f(u);y];z] \qquad (\operatorname{Re}\rho>0,\ \operatorname{Re}\nu>0). \qquad (20)\end{aligned}$$

Proof. In Eq. (15) we set

$$g(y)=y^{\nu-1}e^{-zy} \qquad (21)$$

so that

$$\mathscr{L}[g(y);x]=\frac{\Gamma(\nu)}{(x+z)^\nu}. \qquad (22)$$

Now (20) follows immediately by substituting (21) and (22) into (15) and making use of the definition (5) of the generalized Stieltjes transform.

COROLLARY. *Let $g_1(x)$ and its derivatives of orders up to and including $n-1$ be continuous for $x>0$, with $g_1(x)$ and its derivatives having limits at $x=0$, and suppose $g_1(x)$ and its derivatives are of exponential order. Then*

$$\int_0^\infty x^{\rho-1}\mathscr{L}[f(u);x]\{x^n\mathscr{L}[g_1(y);x]-x^{n-1}g_1(0)-\cdots-g_1^{(n-1)}(0)\}\,dx$$

$$=\Gamma(\rho)\int_0^\infty g_1^{(n)}(y)\,\mathscr{S}_\rho[f(u);y]\,dy \qquad (\operatorname{Re}\rho>0). \tag{23}$$

Proof. In Eq. (15) we set

$$g(y)=g_1^{(n)}(y) \tag{24}$$

so that

$$\mathscr{L}[g(y);x]=x^n\mathscr{L}[g_1(y);x]-x^{n-1}g_1(0)-\cdots-g_1^{n-1}(0), \tag{25}$$

by the derivative property of the Laplace transform. Now the result (23) follows from (24), (25), and (15).

APPLICATIONS OF THE MAIN THEOREM

A consequence of the main theorem involving the Widder potential transform, the Laplace transform, the generalized Stieltjes transform and the Fourier sine and cosine transforms may be stated as the following:

THEOREM. *We have*

$$\mathscr{P}[x^{\rho-1}\mathscr{L}[f(u);x];z]=\Gamma(\rho)\,\mathscr{F}_c[\mathscr{S}_\rho[f(u);y];z] \qquad (\operatorname{Re}\rho>0), \tag{25'}$$

$$\mathscr{P}[x^{\rho-2}\mathscr{L}[f(u);x];z]=\frac{\Gamma(\rho)}{z}\,\mathscr{F}_s[\mathscr{S}_\rho[f(u);y];z] \qquad (\operatorname{Re}\rho>0). \tag{26}$$

Proof. In Eq. (15) we set

$$g(y)=\cos(zy) \tag{27}$$

so that

$$\mathscr{L}[g(y);x]=\frac{x}{x^2+z^2}. \tag{28}$$

Now the result (25′) follows from (27), (28), and (15). The proof of (26) is similar.

A consequence of Eq. (23) involving the generalized Stieltjes transform, the Laplace transform, and the Laguerre transform may be stated as follows:

THEOREM. *We have*

$$\mathscr{S}_{n+\alpha+1}[x^{n+\rho-1}\mathscr{L}[f(u);x];1] = \frac{n!\,\Gamma(\rho)}{\Gamma(n+\alpha+1)}\mathscr{T}_\alpha[\mathscr{S}_\rho[f(u);y];n] \quad (n=0,1,2,\ldots,\alpha>-1,\operatorname{Re}\rho>0), \tag{29}$$

$$\mathscr{L}[y^{n+\alpha}\mathscr{S}_{n+\alpha}[f(u);y];1] = \frac{n!\,\Gamma(\rho)}{\Gamma(n+\alpha)}\mathscr{T}_\alpha[\mathscr{S}_\rho[f(u);y];n] \quad (n=1,2,\ldots,\alpha>-1,\operatorname{Re}\rho>0). \tag{30}$$

Proof. In Eq. (23) we set

$$g_1(y)=y^{n+\alpha}e^{-y} \tag{31}$$

so that

$$\mathscr{L}[g_1(y);x]=\frac{\Gamma(n+\alpha+1)}{(x+1)^{n+\alpha+1}} \tag{32}$$

and

$$g_1^k(0)=0, \qquad \text{for} \quad 0\leqslant k\leqslant n-1. \tag{33}$$

Substituting (31), (32), and (33) into (23) we obtain

$$\begin{aligned}\int_0^\infty \frac{x^{n+\rho-1}}{(x+1)^{n+\alpha+1}}\mathscr{L}[f(u);x]\,dx &= \frac{\Gamma(\rho)}{\Gamma(n+\alpha+1)}\int_0^\infty \frac{d^n}{dy^n}(e^{-y}y^{n+\alpha})\,\mathscr{S}_\rho[f(u);y]\,dy\\ &= \frac{n!\,\Gamma(\rho)}{\Gamma(n+\alpha+1)}\int_0^\infty e^{-y}y^\alpha\mathscr{L}_n^{(\alpha)}(y)\,\mathscr{S}_\rho[f(u);y]\,dy.\end{aligned} \tag{34}$$

Now the assertion (29) follows from (5), (9), (10), and (34). The assertion (30) easily follows from (20).

SOME ILLUSTRATIVE EXAMPLES

A simple illustration of the main theorem is provided by the following:

EXAMPLE 1. We show that

$$\mathscr{M}[e^{ay}\Gamma(1-\rho, ay); \mu] = \frac{\pi\Gamma(\mu)\, z^{\rho-\mu-1}}{\Gamma(\rho)\sin\pi(\rho-\mu)} \qquad (\operatorname{Re}\mu+1>\operatorname{Re}\rho>\operatorname{Re}\mu>0,\ \operatorname{Re}a>0), \tag{35}$$

where $\Gamma(a, x)$ is the incomplete gamma function.

In Eq. (15) of the main theorem we set

$$f(u)=e^{-au} \qquad \text{and} \qquad g(y)=y^{\mu-1} \tag{35'}$$

so that

$$\mathscr{L}[f(u); x]=\frac{1}{(x+a)} \qquad \text{and} \qquad \mathscr{L}[g(y); x]=\frac{\Gamma(\mu)}{x^{\mu}}. \tag{36}$$

From (12) and the formula [1, Vol. I, p. 137, (7)] we obtain

$$\mathscr{S}_{\rho}[f(u); x]=\frac{1}{\Gamma(\rho)}\mathscr{L}\left[\frac{x^{\rho-1}}{x+a}; y\right]=a^{\rho-1}e^{ay}\Gamma(1-\rho, ay). \tag{37}$$

Substituting (35′), (36), and (37) into (15) we obtain

$$\int_0^{\infty}\frac{x^{\rho-\mu-1}}{x+a}\,dx=\frac{\Gamma(\rho)}{\Gamma(\mu)}a^{\rho-1}\int_0^{\infty}x^{\mu-1}e^{ay}\Gamma(1-\rho, ay)\,dy. \tag{38}$$

From the definitions (4) and (11) of the Stieltjes transform and the Mellin transform, respectively, we obtain

$$\mathscr{M}[e^{ay}\Gamma(1-\rho, ay); \mu]=\frac{\Gamma(\mu)}{\Gamma(\rho)}a^{1-\rho}\mathscr{S}[x^{\rho-\mu-1}; a]. \tag{39}$$

Making use of (12), the Stieltjes transform on the right-hand side of (39) may be evaluated and our assertion (35) then follows immediately.

EXAMPLE 2. We show that

$$\mathscr{T}_{\alpha}[y^{(1/2)(\nu-\rho+1)}K_{\nu-\rho+1}(ay^{1/2}); n] = \frac{a^{n+\alpha-\rho-1}}{2^{n+\alpha-\rho}}\frac{\Gamma(\nu+2)\,\Gamma(\rho)\,\Gamma(n+\alpha+1)}{n!}e^{a^2/8}W_{-k,l}\left(\frac{a^2}{4}\right), \tag{40}$$

where $a>0$, $\operatorname{Re}\rho>0$, $\operatorname{Re}\nu>-1$, $\operatorname{Re}\alpha+n>0$, $2k=n+\alpha+\nu+2$, $2l=\nu-n-\alpha+1$, and $n=1, 2, ...,$ $K_\nu(x)$ denotes the modified Bessel function of the third kind and $W_{k,l}(x)$ denotes Whittaker's confluent hypergeometric function.

In Eq. (30) we set $f(u)=u^{\nu/2}J_\nu(au^{1/2})$ so that (cf. [1, Vol. II, p. 235, (20) and Vol. I, p. 199, (37)])

$$\mathscr{S}_\rho[f(u); y]=\frac{a^{\rho-1}}{2^\rho\Gamma(\rho)}\, y^{(1/2)(\nu-\rho+1)}K_{\nu-\rho+1}(ay^{1/2}) \tag{41}$$

and

$$\mathscr{L}[y^{n+\alpha}\mathscr{S}_{n+\alpha}[f(u); y]; 1] = \frac{a^{n+\alpha-2}}{2^{n+\alpha}}\,\frac{\Gamma(\nu+2)\,\Gamma(n+\alpha+1)}{\Gamma(n+\alpha)}\,e^{a^2/8}W_{-k,l}\left(\frac{a^2}{4}\right), \tag{42}$$

where k and l are as defined in the statement of the example. Substituting (41) and (42) into (30) we obtain the assertion (40).

EXAMPLE 3. We show that

$$\mathscr{L}[y^{\mu+l-3/2}e^{ay/2}W_{k,l}(ay); b] = \frac{a^{1/2-l}}{b^\mu}\,\frac{\Gamma(\mu-\rho+\lambda)\,\Gamma(\mu)}{\Gamma(\mu+\lambda)}\,{}_2F_1\left(\mu, \rho; \mu+\lambda; 1-\frac{a}{b}\right), \tag{43}$$

where $a>0$, $b>0$, $\operatorname{Re}\rho>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}\lambda>0$, $\operatorname{Re}\lambda>\operatorname{Re}(\rho-\mu)$, $2k=1-\lambda-\rho$, $2l=\lambda-\rho$, ${}_2F_1$ denotes Gauss's hypergeometric function, and $W_{k,l}(x)$ denotes Whittaker's hypergeometric function.

In Eq. (15) we set

$$f(u)=u^{\lambda-1}e^{-au} \qquad \text{and} \qquad g(y)=y^{\mu-1}e^{-by} \tag{44}$$

so that

$$\mathscr{L}[f(u); x]=\frac{\Gamma(\lambda)}{(x+a)^\lambda}, \qquad \mathscr{L}[g(y); x]=\frac{\Gamma(\mu)}{(x+b)^\mu} \tag{45}$$

and (cf. [1, Vol. II, p. 234, (12)])

$$\mathscr{S}_\rho[f(u); x]=\Gamma(\lambda)\, a^{-l-1/2}y^{l-1/2}e^{ay/2}W_{k,l}(ay), \tag{46}$$

where l and k are as in the statement of the example. Substituting (44), (45), and (46) into (15) and using the definitions (2) and (5) of the Laplace transform and the generalized Stieltjes transform, respectively, we obtain

$$\mathcal{S}_{\lambda}\left[\frac{x^{\rho-1}}{(x+b)^{\mu}};a\right]=a^{-l-1/2}\frac{\Gamma(\rho)}{\Gamma(\mu)}\mathcal{L}[y^{\mu+l-3/2}e^{ay/2}W_{k,l}(ay);b]. \tag{47}$$

Now the assertion (43) follows from (47) and the formula in [1, Vol. II, p. 233, (9)].

We conclude by remarking that many other infinite integrals can be evaluated in this manner by applying the main theorem and its corollaries considered above.

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References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Tables of Integral Transforms," Vols. I and II, McGraw-Hill, New York, 1954.
2. S. Goldstein, Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function, *Proc. London Math. Soc. (2)* **34** (1932), 103–125.
3. H. M. Srivastava and R. Panda, Certain multidimensional integral transformations, I and II, *Nederl. Akad. Wetensch. Indag. Math.* **40** (1978), 118–131, 132–143.
4. H. M. Srivastava and S. P. Singh, A note on the Widder transform related to the Poisson integral for a half-plane, *Internat. J. Math. Ed. Sci. Tech.* **16** (1985), 675–677.
5. H. M. Srivastava and O. Yürekli, A theorem on Widder's potential transform and its applications, *J. Math. Anal. Appl.* **154** (1991), 585–593.
6. D. V. Widder, A transform related to the Poisson integral for a half-plane, *Duke Math. J.* **33** (1966), 355–362.
7. D. V. Widder, "An Introduction to Transform Theory," Academic Press, New York, 1971.
8. O. Yürekli, A Parseval-type them applied to certain integral transforms, *J. Appl. Math.* **42** (1989), 241–249.
9. O. Yürekli and I. Sadek, A Parseval–Goldstein type theorem on the Widder potential transform and its applications, *Internat. J. Math. Math. Sci.* **14** (1991), 517–524.